The backward energy cascade is modulated by controlling the sign and amplitude of the product $f(I_{\rm LS})I_{\rm LS}$. The authors considered a number of choices. The first is:

$$f(I_{\rm LS}) = \begin{cases} 1 & \text{if} \quad I_{\rm LS} \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad . \tag{6.102}$$

This solution makes it possible to cancel out the representation of the backward cascade completely by forcing the model to be strictly dissipative. One drawback to this is that the function f is discontinuous, which can generate numerical problems. A second solution that is continuous consists in taking:

$$f(I_{\rm LS}) = \begin{cases} I_{\rm LS} & \text{if} \quad I_{\rm LS} \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad . \tag{6.103}$$

One last positive, continuous, upper-bounded solution is of the form:

$$f(I_{\rm LS}) = \begin{cases} (1 - \exp(-\gamma I_{\rm LS}^2)) & \text{if} \quad I_{\rm LS} \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad , \tag{6.104}$$

in which $\gamma = 10$.

Dynamic Similarity Model. A dynamic version of the Liu–Meneveau–Katz model (6.95) was also proposed [200] for which the constant $C_{\rm l}$ will no longer be set arbitrarily. To compute this model, we introduce a third level of filtering identified by $\widehat{\cdot}$. The Q analogous to tensor $\mathcal{L}^{\rm m}$ for this new level of filtering is expressed:

$$Q_{ij} = (\widehat{\widetilde{\overline{u}}_i \widetilde{\overline{u}}_j} - \widehat{\widetilde{\overline{u}}}_i \widehat{\widetilde{\overline{u}}}_j) \quad . \tag{6.105}$$

The Germano–Lilly dynamic procedure, based here on the difference:

$$M_{ij} = f(I_{\rm QS}) Q_{ij} - f(\widehat{I_{\rm LS}}) \mathcal{L}_{ij}^{\rm m} \quad , \tag{6.106}$$

where

$$I_{\rm QS} = \frac{Q_{mn} \widetilde{\overline{S}}_{mn}}{|Q|\,|\widetilde{\overline{S}}|} \quad , \tag{6.107}$$

yields:

$$C_{\rm l} = \frac{\mathcal{L}_{lk}^{\rm m} M_{lk}}{M_{pq} M_{pq}} \quad . \tag{6.108}$$

6.4.3 A Bridge Between Scale Similarity and Approximate Deconvolution Models. Generalized Similarity Models

The Bardina model can be interpreted as a particular case of the approximate deconvolution based models described in Sect. 6.1.

Using the second order differential approximation

$$\overline{\phi} = \phi + \frac{\alpha^{(2)}}{2} \frac{\partial^2 \phi}{\partial x^2} \quad , \tag{6.109}$$

the Bardina model (6.92) is strictly equivalent to the second order gradient model given by relations (6.13) and (6.14).

It can also be derived using the iterative deconvolution procedure: a zeroth-order truncation in (6.27) is used to recover relation (6.88), while a first-order expansion is employed to derive (6.89).

The Bardina model then appears as a low-order formal expansion model for the subgrid tensor. Generalized scale similarity models can then be defined using higher-order truncations for the formal expansion [119]. They are formulated as

$$\tau_{ij} = \overline{(G_d^{-1} \star \overline{u}_i)(G_d^{-1} \star \overline{u}_j)} - \overline{(G_d^{-1} \star \overline{u})}_i - \overline{(G_d^{-1} \star \overline{u})}_j \quad , \tag{6.110}$$

where $G_d^{-1}\star$ designates the approximate deconvolution operator, defined using equation (6.9) or equation (6.27).

6.5 Mixed Modeling

6.5.1 Motivations

The structural models based on the scale similarity idea, and the functional models, each have their advantages and disadvantages that make the seem complementary:

- The functional models, generally, correctly take into account the level of the energy transfers between the resolved scales and the subgrid modes. However, their prediction of the subgrid tensor structure, *i.e.* its eigenvectors, is very poor.
- The models based on the scale-similarity hypothesis or an approximate deconvolution procedure generally predict well the structure of the subgrid tensor better (and then are able to capture anisotropic effects and disequilibrium), but are less efficient for dealing with the level of the energy transfers.

Tests have shown that they are able to capture disequilibrium and anisotropy effects.

Shao *et al.* [308] propose a splitting of the kinetic energy transfer across the cut-off that enlights the role of each one of these two model classes. These authors combine the classical large-eddy simulation convolution filter to the ensemble average, yielding the following decompositions:

$$\begin{aligned} \mathbf{u} &= \langle \mathbf{u} \rangle + \mathbf{u}'^e && (6.111)\\ &= \overline{\mathbf{u}} + \mathbf{u}' && (6.112)\\ &= \langle \overline{\mathbf{u}} \rangle + \langle \mathbf{u}' \rangle + \overline{\mathbf{u}'^e} + \mathbf{u}'' \quad . && (6.113) \end{aligned}$$

Using this hybrid decomposition, the subgrid tensor splits into

$$\tau_{ij} = \tau_{ij}^{\text{rapid}} + \tau_{ij}^{\text{slow}} \quad , \tag{6.114}$$

with

$$\tau_{ij}^{\text{slow}} = \overline{u_i'^e u_j'^e} - \overline{u'^e}_i \overline{u'^e}_j \quad , \tag{6.115}$$

$$\begin{aligned} \tau_{ij}^{\text{rapid}} = {} & \overline{\langle u_i \rangle \langle u_j \rangle} - \langle \overline{u}_i \rangle \langle \overline{u}_j \rangle + \overline{u_i'^e \langle u_j \rangle} - \overline{u'^e}_i \langle \overline{u}_j \rangle \\ & + \overline{u_j'^e \langle u_i \rangle} - \overline{u'^e}_j \langle \overline{u}_i \rangle \quad . \end{aligned} \tag{6.116}$$

These two parts can be analysed as follows:

- The rapid part explicitly depends on the mean flow. This contribution arises only if the convolution filter is applied in directions where the mean flow gradients are non-zero. It is referred to as rapid because the time scale of its response to variations of the mean flow is small. Numerical experiments show that this part plays an important role when the turbulence is in a desiquilibrium state when: (i) production of kinetic energy is much larger than dissipation or (ii) the filter length is of the same order as the integral scale of turbulence. Subgrid stresses anisotropy is observed to be due to the interaction of this rapid part and the mean shear. Numerical simulations have shown that the rapid part escapes the functional modeling, but scale-similarity models succeed in representing anisotropic energy transfer (both forward and backward cascades) associated to the rapid part.
- The slow part is always present in large-eddy simulation, because it does not depend on the mean flow gradients. It corresponds to the subgrid tensor analyzed through the previously described canonical analysis. It is referred to as slow because its relaxation time is long with respect to rapid part. Numerical tests show that subgrid viscosity model correctly capture the associated kinetic energy transfer.

One simple idea for generating subgrid models possessing good qualities on both the structural and energy levels is to combine a functional with a structural model, making what is called mixed models. This is generally done by combining a subgrid viscosity model for representing the energy cascade mechanism with a scale similarity. The stochastic backward cascade models are usually not included because the structural models are capable of including this phenomenon. The resulting form is

$$\tau_{ij} - \frac{1}{3}\tau_{kk}\delta_{ij} = -2\nu_{\text{sgs}}\overline{S}_{ij} + (L_{ij} - \frac{1}{3}L_{kk}\delta_{ij}) \quad , \tag{6.117}$$

where ν_{sgs} is the subgrid viscosity (evaluated using one of the previously described model), and L_{ij} the evaluation obtained using one of the structural model[12].

Examples of such models are described in the following.

6.5.2 Examples of Mixed Models

We present several examples of mixed models here:

1. The Smagorinsky–Bardina model (p.185), for which the respective weights of each of the contributions are preset. This model is limited by the hypotheses underlying each of the two parts constituting it: the subgrid viscosity is still based on arguments of the infinite inertial range type. Experience shows, though, that combining the two models reduces the importance of the constraints associated with these underlying hypotheses, which improves the results.
2. A one-parameter mixed model whose subgrid viscosity is computed by a dynamic procedure of the Germano–Lilly type (p.186). With this procedure, the respective weights of the structural and functional parts of the model can be modified, so that the subgrid viscosity model is now computed as a complement to the scale similarity model, which allows a better control of the dissipation induced. It can be said, though, that this procedure innately prefers the structural part.
3. The general form of N-parameter dynamic mixed model, as derived by Sagaut *et al.* (p.187). This procedure is an extension of the previous one: the weights of the different parts of the model are dynamically computed, resulting in a possibly better approximation of the true subgrid stresses. The case of two-parameter dynamic mixed model is emphasized.

Mixed Smagorinsky–Bardina Model. The first example is proposed by Bardina *et al.* [13] in the form of a linear combination of the Smagorinsky model (4.90) and the scale similarity model (6.92). The subgrid tensor deviator is then written:

[12] Only scale-similarity models or approximate deconvolution models are used in practice to derive mixed models, because they are very easy to implement.

$$\tau_{ij} - \frac{1}{3}\tau_{kk}\delta_{ij} = \frac{1}{2}\left(-2\nu_{\text{sgs}}\overline{S}_{ij} + \mathcal{L}_{ij} - \frac{1}{3}\mathcal{L}_{kk}\delta_{ij}\right) \quad , \tag{6.118}$$

in which

$$\mathcal{L}_{ij} = \left(\overline{\overline{u}_i\,\overline{u}_j} - \overline{\overline{u}}_i\,\overline{\overline{u}}_j\right) \quad , \tag{6.119}$$

and

$$\nu_{\text{sgs}} = C_{\text{s}}\overline{\Delta}^2|\overline{S}| \quad . \tag{6.120}$$

Variants are obtained either by changing the subgrid viscosity model used or by replacing the tensor $\mathcal{L}$ with the tensor $\mathcal{L}^{\text{m}}$ (6.98) or the tensor $\overline{\mathcal{L}}$ (6.94).

One-Parameter Mixed Dynamic Model. A mixed dynamic modeling was proposed by Zang, Street, and Koseff [366]. This is based initially on the Bardina model coupled with the Smagorinsky model, but the latter can be replaced by any other subgrid viscosity model. The subgrid viscosity model constant is computed by a dynamic procedure. The subgrid tensors corresponding to the two filtering levels are modeled by a mixed model:

$$\tau_{ij} - \frac{1}{3}\tau_{kk}\delta_{ij} = -2\nu_{\text{sgs}}\overline{S}_{ij} + \mathcal{L}^{\text{m}}_{ij} - \frac{1}{3}\mathcal{L}^{\text{m}}_{kk}\delta_{ij} \quad , \tag{6.121}$$

$$T_{ij} - \frac{1}{3}T_{kk}\delta_{ij} = -2\nu_{\text{sgs}}\widetilde{\overline{S}}_{ij} + \mathcal{Q}_{ij} - \frac{1}{3}\mathcal{Q}_{kk}\delta_{ij} \quad , \tag{6.122}$$

in which

$$\mathcal{Q}_{ij} = \widetilde{\overline{\overline{u}_i\overline{u}_j}} - \widetilde{\overline{\overline{u}}}_i\widetilde{\overline{\overline{u}}}_j \quad , \tag{6.123}$$

and

$$\nu_{\text{sgs}} = C_{\text{d}}\overline{\Delta}|\overline{S}| \quad . \tag{6.124}$$

The residual E_{ij} is now of the form:

$$E_{ij} = \mathcal{L}^{\text{m}}_{ij} - \mathcal{H}_{ij} - \left(-2C_{\text{d}}\overline{\Delta}^2 m_{ij} + \delta_{ij}P_{kk}\right) \quad , \tag{6.125}$$

in which

$$\mathcal{H}_{ij} = \widetilde{\overline{\overline{u}_i\overline{u}_j}} - \widetilde{\overline{\overline{u}}}_i\widetilde{\overline{\overline{u}}}_j \quad , \tag{6.126}$$

$$\mathcal{L}^{\text{m}}_{ij} = \widetilde{\overline{u}_i\overline{u}_j} - \widetilde{\overline{u}}_i\widetilde{\overline{u}}_j \quad , \tag{6.127}$$

$$m_{ij} = \left(\frac{\widetilde{\overline{\Delta}}}{\overline{\Delta}}\right)^2 |\widetilde{\overline{S}}|\widetilde{\overline{S}}_{ij} - \widetilde{|\overline{S}|\overline{S}_{ij}} \quad , \tag{6.128}$$

and where P_{kk} represents the trace of the subgrid tensor. The Germano–Lilly dynamic procedure leads to:

$$C_{\text{d}} = \frac{(\mathcal{L}^{\text{m}}_{ij} - \mathcal{H}_{ij})m_{ij}}{m_{ij}m_{ij}} \quad . \tag{6.129}$$

In simulations performed with this model, the authors observed a reduction in the value of the dynamic constant with respect to that predicted by the usual dynamic model (*i.e.* based on the Smagorinsky model alone). This can be explained by the fact that the difference between the $\mathcal{L}^{\text{m}}$ and $\mathcal{H}$ terms appears in the numerator of the fraction (6.129) and that this difference is small because these terms are very similar. This shows that the subgrid viscosity model serves only to model a residual part of the full subgrid tensor and not its entirety, as in the usual dynamic model.

Vreman *et al.* [341] propose a variant of this model. For the sake of mathematical consistency, by making the model for the tensor T_{ij} dependent only on the velocity field that corresponds to the same level of filtering, *i.e.* $\widetilde{\overline{\mathbf{u}}}$, these authors propose the following alternate form for the tensor $\mathcal{Q}_{ij}$:

$$\mathcal{Q}_{ij} = \widetilde{\widetilde{\overline{\overline{u}_i\overline{u}_j}}} - \widetilde{\widetilde{\overline{\overline{u}}}}_i\widetilde{\widetilde{\overline{\overline{u}}}}_j \quad . \tag{6.130}$$

N-Parameter Dynamic Mixed Model

General Formulation and Formal Resolution. A general form of multiparameter dynamic model was derived by Sagaut *et al.* [288]. Considering a formal N-part parametrization of the subgrid tensor, each term being associated to a real constant $C_l, k = 1, .., N$

$$\tau_{ij} = \sum_{l=1,N} C_l f^l_{ij}(\overline{\mathbf{u}}, \overline{\Delta}) \quad , \tag{6.131}$$

where the functions f^l_{ij} are the kernels of the different parts of the complete model. The equivalent formulation obtained at the test filter level is

$$T_{ij} = \sum_{l=1,N} C_l f^l_{ij}(\widetilde{\overline{\mathbf{u}}}, \widetilde{\overline{\Delta}}) \quad . \tag{6.132}$$

Inserting (6.131) and (6.132) into the Germano identity (4.126), we get the following definition of the residual E_{ij}:

$$E_{ij} = L_{ij} - \sum_{l=1,N} C_l m^l_{ij}, \;\; m^l_{ij} = f^l_{ij}(\widetilde{\overline{\mathbf{u}}}, \widetilde{\overline{\Delta}}) - \widetilde{f^l_{ij}}(\overline{\mathbf{u}}, \overline{\Delta}) \quad . \tag{6.133}$$

In order to obtain N linearly independent relations to compute the constants C_l, a first solution is to operate the contraction of the residual (6.133) with N independent tensors A^l_{ij}. The constants will then appear as the solutions of the follow linear algebraic problem of rank N: